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EIGENFREQUENCIES OF LINE SUPPORTED RECTANGULAR PLATES

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Abstract—A set of series consisting of a combination of free vibrating beam functions and polynomials is used as the admissible functions in the Rayleigh–Ritz method to study the problem of the flexural vibration of thin, uniform thickness, orthotropic rectangular plates which may be continuous over several supports in one or two directions. In practice, which kind of free vibrating beam functions are used is determined by the boundary conditions of the plate. The starting order of the beam eigenfunctions and the number of terms in the polynomials in each direction depend on the number of intermediate line supports in that direction. The coefficients of the polynomials are decided by the boundary conditions of the plate and the locations of the line supports. Some numerical results are given for the simply-supported and/or clamped rectangular plates with two- and three-spans in one and two directions. It is demonstrated that the method may be used to tackle such plate problems and has considerable accuracy and fast convergence compared with the available results, the difficulty of application not being greater than any other methods.

1. INTRODUCTION

In aeronautical, civil and naval engineering and the like, many structures may be simplified to rectangular plates with intermediate supports in one or two directions. It is of great importance to study the dynamic characteristics of such plates. In mechanical analysis, the intermediate supports of continuous plates may be treated as rigid line supports with respect to lateral translation, offering no rotational restraint and/or stringers including the effect of translational and rotational rigidity and inertia of the supports. In the present paper, attention will be confined to the study of plates continuous over line supports in one and two directions.

Much of the work reported in the literature has been concerned with plates continuous only in one direction. Early research mainly focused on the rectangular plates simply supported at two opposite edges and continuous over rigid supports perpendicular to those edges. Ungar (1961) treated a two-span, simply-supported plate and presented a semi-graphical approach. Veletsos and Newmark (1956) used the Holzer method for torsional vibration of shafts to determine the eigenfrequencies of plates simply supported along the continuous edges. The transfer matrix method was developed by Mercer and Seavey (1967) for analysis of such plates. Moskalenko (1969) proposed an orthogonalization method of finite-difference equations in vibration analysis of the periodically-supported plates with a two-span period. The effect of the boundary conditions along the edges perpendicular to the periodic supports was analysed by Bolotin (1961a) and by Moskalenko and Chien De-lin (1965) under Bolotin's edge effect method (1961b), two- and three-span plates were considered respectively. Dickinson and Warburton (1967) also used Bolotin's method for the study of two-span plates involving clamped, simply-supported and free edges. The modified Bolotin method, developed by Vijayakumar (1971) and Elishakoff (1974) was used by Elishakoff and Sternberg (1979) for the determination of the eigenfrequencies of rectangular plates continuous over line supports with an arbitrary number of equal spans in one direction. More recently, the receptance method was used by Azimi *et al.* (1984) for three- and four-span plates in one direction. Cheung and Cheung (1971) studied the effect of the boundary conditions at the edges perpendicular to the periodic supports through the finite-strip method.

The problem of the vibration of line-supported plates which are continuous in two directions has received rather less attention. However, Dill and Pister (1958) presented a series solution to analyse plates continuous in one and two directions. The series solution was extended by Dickinson (1969, 1971) and found an application in orthotropic plates and plate systems. Wu and Cheung (1974) used a method of finite elements in conjunction with Bolotin's approach to plates continuous in two directions. Lindberg and Olson (1967) applied the finite element method for all-round clamped multispan plates. Tokahashi and Chishaki (1979) presented a sine series solution for the simply-supported plates continuous over a number of line supports in two directions. In a recent paper by Kim and Dickinson (1987), a set of orthogonal polynomials which is developed from Bhat's functions (1985) generated by a recurrence formula, was proposed as the admissible functions for use in the Rayleigh–Ritz method to study the flexural vibration of line-supported rectangular plates and plate systems. A combination of free vibrating beam functions and polynomials was used to study buckling and vibration of continuous folded plates by Cheung and Delcourt (1977), Delcourt and Cheung (1978) and Cheung and Swaddiwudhipong (1982).

Although there are several approaches to calculating the vibration characteristics of plates with line supports in one and two directions, it is of great significance to provide what are often simpler, more efficient and/or more accurate solutions to those problems for which such exist. In this paper, a set of series consisting of a combination of beam eigenfunctions and polynomials is selected as the admissible functions of line-supported plates. The formulation of the Rayleigh–Ritz method when using the admissible functions suggested by this paper is straightforward for continuous plate problems and yields results of comparable accuracy with no computational difficulty. The analytical procedure is outlined in this paper and some numerical results are given for two- and three-span rectangular plates with the edges simply supported and/or clamped in one and two directions. In several instances, comparisons are made with values available in the literature and in all cases, close agreement may be seen to be achieved. As seen from the results, the convergence is very fast and the accuracy is generally better than Kim and Dickinson's (1987).

2. MATHEMATICAL MODEL

It is assumed that the plate under consideration lies in the x–y plane, is bound by edges $x = 0$, $x = a$ and $y = 0$, $y = b$ and is of uniform thickness, rectangularly orthotropic material with principal axes orthogonal to the edges. The intermediate line supports are also assumed to lie orthogonal to the plate edges and to the present motion in the z-direction but to offer no resistance to normal rotation. A constant in-plane force can be included in the analysis without any difficulty but, for the sake of brevity, is omitted here.

From the vibrational theory of thin plates, the strain and kinetic energies of the elastic plate in Cartesian co-ordinates are as follows:

$$U = \frac{1}{2}\int_0^a \int_0^b \left\{ D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2H\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 4D_{xy}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} \mathrm{d}y\,\mathrm{d}x, \quad (1a)$$

$$T = \tfrac{1}{2}\rho h \int_0^a \int_0^b \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}y\,\mathrm{d}x, \quad (1b)$$

where w is the deflection of the plate in the z-direction and ρ is the material density. The quantities $D_x = E_x h^3/12(1-\nu_{xy}\nu_{yx})$, $D_y = D_x E_y/E_x$, $D_{xy} = G_{xy}h^3/12$ and $H = \nu_{xy}D_y + 2D_{xy}$ are flexural rigidities, in which E_x and E_y are Young's moduli in the x- and y- directions, respectively, G_{xy} is the shear modulus, ν_{xy} and ν_{yx} are Poisson's ratios and h is the plate thickness. This equation also applies to the isotropic case, for which $\nu_{xy} = \nu_{yx} = \nu$, $D_x = D_y = H = Eh^3/12(1-\nu^2) = D$ and $D_{xy} = (1-\nu)D/2$.

From the Hamiltonian principle, the stationary value exists for the true solution, i.e. there is the following variation

$$\delta(U-T) = 0. \tag{2}$$

For free vibration of the plate, the deflection w may be expressed as

$$W(x, y, t) = W(x, y)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{3}$$

where ω is the radian natural frequency of vibration and t is time, $\mathrm{i} = \sqrt{-1}$. Assuming the variables in $W(x, y)$ are separable, mode shape function $W(x, y)$ may be expressed in terms of series as

$$W(x, y) = \sum_m \sum_n A_{mn}\varphi_m(x)\psi_n(y), \tag{4}$$

where $\varphi_m(x)$ and $\psi_n(y)$ are appropriate admissible functions which satisfy at least the geometrical boundary conditions and, if possible, all the boundary conditions. A_{mn} are unknown coefficients. Substituting eqns (3) and (4) into eqn (1) and making use of eqn (2) to minimize with respect to the coefficients A_{mn} leads to the eigenvalue equation

$$\sum_m \sum_n [C_{mnij} - \lambda E_{mi}^{(0,0)} F_{nj}^{(0,0)}]A_{mn} = 0, \tag{5}$$

where

$$C_{mnij} = \frac{D_x}{H} E_{mi}^{(2,2)} F_{nj}^{(0,0)} \frac{b^2}{a^2} + \frac{D_y}{H} E_{mi}^{(0,0)} F_{nj}^{(2,2)} \frac{a^2}{b^2} + E_{mi}^{(0,2)} F_{nj}^{(2,0)} + E_{mi}^{(2,0)} F_{nj}^{(0,2)}$$

$$+ 2\frac{D_{xy}}{H}\{2E_{mi}^{(1,1)} F_{nj}^{(1,1)} - E_{mi}^{(0,2)} F_{nj}^{(2,0)} - E_{mi}^{(2,0)} F_{nj}^{(0,2)}\}, \quad m, n, i, j = 1, 2, 3, \ldots,$$

$$\lambda = \rho h \omega^2 a^2 b^2 / H, \quad E_{mi}^{(r,s)} = a^{(r+s-1)} \int_0^a (\mathrm{d}^r\varphi_m/\mathrm{d}x^r)(\mathrm{d}^s\varphi_i/\mathrm{d}x^s)\,\mathrm{d}x,$$

$$F_{nj}^{(r,s)} = b^{(r+s-1)} \int_0^b (\mathrm{d}^r\psi_n/\mathrm{d}y^r)(\mathrm{d}^s\psi_j/dy^s)\,\mathrm{d}y, \quad r, s = 0, 1, 2.$$

The solution of eqn (5) yields the natural frequencies of vibration of the plate together with the coefficients for the mode shapes (4). From the above analysis the validity and accuracy of this solution completely depend on the choice of the admissible functions $\varphi_m(x)$ and $\psi_n(y)$. There have been several approaches to selecting $\varphi_m(x)$ and $\psi_n(y)$. However, the simplicity, convergence and accuracy are not always satisfying. Taking account of the structural characteristics of the rectangular plates with intermediate line supports, here the admissible functions are taken as follows:

$$\varphi_m(x) = X_{m+lx}(x) + \sum_{k=0}^{lx+3} C_{mk}x^k = X_{m+lx}(x) + \tilde{X}_m(x), \quad m = 1, 2, 3, \ldots, \tag{6a}$$

$$\psi_n(y) = Y_{n+ly}(y) + \sum_{k=0}^{ly+3} D_{nk}y^k = Y_{n+ly}(y) + \tilde{Y}_n(y), \quad n = 1, 2, 3, \ldots, \tag{6b}$$

where lx and ly are the numbers of the intermediate line supports running perpendicular to the x- and y-axes, respectively. C_{mk} and D_{nk} are unknown coefficients of the polynomials which may be decided by the boundary conditions and the zero deflection conditions at the intermediate line supports. $X_{m+lx}(x)$ and $Y_{n+ly}(y)$ are the free vibrating beam eigenfunctions satisfying the corresponding boundary conditions of the plate in the x- and y-directions, respectively. It is of great importance to notice that the starting orders of the beam eigenfunctions in eqns (6a) and (6b) are not from the first but depend on the number of the intermediate line supports in that direction and the numbers of the terms of polynomials are also decided by the number of the line supports in that direction.

From the vibrational theory of beams, the vibrating beam eigenfunctions may be easily given for arbitrary boundary conditions. For example, for a free vibrating clamped–clamped beam, the eigenfunctions are

$$X_i(x) = \cosh(k_i x/a) - \cos(k_i x/a) - \alpha_i(\sinh(k_i x/a) - \sin(k_i x/a)),$$
$$Y_i(y) = \cosh(k_i y/b) - \cos(k_i y/b) - \alpha_i(\sinh(k_i y/b) - \sin(k_i y/b)), \quad i = 1, 2, 3, \ldots,$$

where k_i is the zero of the expression

$$\cos k_i \cosh k_i = 1,$$

and where α_i is

$$\alpha_i = (\cos k_i - \cosh k_i)/(\sin k_i - \sinh k_i).$$

For a clamped–free beam, the expressions of the eigenfunctions are the same as those of clamped–clamped beams, but k_i and α_i satisfy, respectively,

$$\cos k_i \cosh k_i = -1, \quad \alpha_i = (\cosh k_i + \cos k_i)/(\sinh k_i - \sin k_i).$$

For a clamped–simply-supported beam, the expresssions of the eigenfunctions and α_i are the same as those of the clamped–clamped beam, but here k_i satisfies

$$\tan k_i = \tanh k_i.$$

For a free vibrating simply-supported–simply-supported beam, the eigenfunctions are

$$X_i(x) = \sqrt{2}\sin(i\pi x/a); \; Y_i(y) = \sqrt{2}\sin(i\pi y/b), \quad i = 1, 2, 3, \ldots.$$

For a free vibrating simply-supported–free beam, the eigenfunctions are

$$X_1(x) = \sqrt{3}(1 - 2x/a),$$
$$X_{i+1}(x) = \cosh(k_i x/a) + \cos(k_i x/a) - \alpha_i(\sinh(k_i x/a) + \sin(k_i x/a)),$$
$$Y_1(y) = \sqrt{3}(1 - 2y/b),$$
$$Y_{i+1}(y) = \cosh(k_i y/b) + \cos(k_i y/b) - \alpha_i(\sinh(k_i y/b) + \sin(k_i y/b)), \quad i = 1, 2, 3, \ldots.$$

For a free vibrating free–free beam, the eigenfunctions are

$$X_1(x) = 1, \quad X_2(x) = \sqrt{3}(1 - 2x/a),$$
$$X_{i+2}(x) = \cosh(k_i x/a) + \cos(k_i x/a) - \alpha_i(\sinh(k_i x/a) + \sin(k_i x/a)),$$
$$Y_1(y) = 1, \quad Y_2(y) = \sqrt{3}(1 - 2y/b),$$
$$Y_{i+2}(y) = \cosh(k_i y/b) + \cos(k_i y/b) - \alpha_i(\sinh(k_i y/b) + \sin(k_i y/b)), \quad i = 1, 2, 3, \ldots.$$

For the last two cases, k_i and α_i are the same as those of the clamped–clamped beam eigenfunctions. The free vibrating beam eigenfunctions for other boundary conditions including elastic supports on edges can be similarly obtained without any difficulty.

Which kind of beam eigenfunctions are used depends on the boundary conditions of the plate. The unknown constants C_{mk} and D_{nk} should be decided by having the admissible functions $\varphi_{m(x)}$ and $\psi_{n(y)}$ to satisfy the boundary conditions of the plate and the zero deflection conditions at the line supports. Since it is quite difficult to construct co-ordinate functions which satisfy the boundary conditions of plates with free edges, it is convenient to replace them approximately, by the corresponding boundary conditions of beams in these cases. Considering the beam eigenfunctions have already satisfied the corresponding boundary conditions, by substituting eqns (6a) and (6b) into the boundary conditions of the plate one has

$$L_{x1}\tilde{X}_m(0) = 0, \quad L_{x2}\tilde{X}_m(0) = 0, \quad L_{x3}\tilde{X}_m(a) = 0,$$
$$L_{x4}\tilde{X}_m(a) = 0, \quad m = 1, 2, 3, \ldots, \tag{7a}$$

$$L_{y1}\tilde{Y}_n(0) = 0, \quad L_{y2}\tilde{Y}_n(0) = 0, \quad L_{y3}\tilde{Y}_n(b) = 0,$$
$$L_{y4}\tilde{Y}_n(b) = 0, \quad n = 1, 2, 3, \ldots, \tag{7b}$$

where L_{xi} and L_{yi} $(i = 1, 2, 3, 4)$ are differential operators which are decided by the boundary conditions of the plate. For example, for the clamped–clamped edges there are

$$L_{x1} = L_{x3} = 1, \quad L_{x2} = L_{x4} = \frac{\mathrm{d}}{\mathrm{d}x}; \quad L_{y1} = L_{y3} = 1, \quad L_{y2} = L_{y4} = \frac{\mathrm{d}}{\mathrm{d}y}.$$

For the clamped–free edges there are

$$L_{x1} = 1, \quad L_{x2} = \frac{\mathrm{d}}{\mathrm{d}x}, \quad L_{x3} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad L_{x4} = \frac{\mathrm{d}^3}{\mathrm{d}x^3};$$
$$L_{y1} = 1, \quad L_{y2} = \frac{\mathrm{d}}{\mathrm{d}y}, \quad L_{y3} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}, \quad L_{y4} = \frac{\mathrm{d}^3}{\mathrm{d}y^3}.$$

For the clamped–simply-supported edges there are

$$L_{x1} = L_{x3} = 1, \quad L_{x2} = \frac{\mathrm{d}}{\mathrm{d}x}, \quad L_{x4} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}; \quad L_{y1} = L_{y3} = 1, \quad L_{y2} = \frac{\mathrm{d}}{\mathrm{d}y}, \quad L_{y4} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}.$$

For the simply-supported–simply-supported edges there are

$$L_{x1} = L_{x3} = 1, \quad L_{x2} = L_{x4} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}; \quad L_{y1} = L_{y3} = 1, \quad L_{y2} = L_{y4} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}.$$

For the simply-supported–free edges, there are

$$L_{x1} = 1, \quad L_{x2} = L_{x3} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad L_{x4} = \frac{\mathrm{d}^3}{\mathrm{d}x^3}; \quad L_{y1} = 1, \quad L_{y2} = L_{y3} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}, \quad L_{y4} = \frac{\mathrm{d}^3}{\mathrm{d}y^3}.$$

For the free–free edges there are

$$L_{x1} = L_{x3} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad L_{x2} = L_{x4} = \frac{\mathrm{d}^3}{\mathrm{d}x^3}; \quad L_{y1} = L_{y3} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}, \quad L_{y2} = L_{y4} = \frac{\mathrm{d}^3}{\mathrm{d}y^3}.$$

Similarly, the differential operators can also be given for other boundary conditions including elastic supports on edges.

From the zero lateral deflection conditions at intermediate line supports, one has

$$\tilde{X}_m(x_k) = -X_{m+lx}(x_k), \quad k = 1, 2, 3, \ldots, lx, \tag{8a}$$
$$\tilde{Y}_n(y_k) = -Y_{n+ly}(y_k), \quad k = 1, 2, 3, \ldots, ly, \tag{8b}$$

where x_k $(k = 1, 2, 3, \ldots, lx)$ and y_k $(k = 1, 2, 3, \ldots, ly)$ are the co-ordinates of the intermediate line supports perpendicular to the x- and y-axes, respectively. From eqns (7) and (8), the coefficients C_{mk} ($k = 0, 1, 2, 3, \ldots, lx+3$) and D_{nk} ($k = 0, 1, 2, 3, \ldots, ly+3$) can be obtained. It may be seen that $\tilde{X}_m(x) = 0$ if there are only intermediate line supports parallel to the x-axis and $\tilde{Y}_n(y) = 0$ if there are only intermediate line supports parallel to the y-axis.

There is orthogonality between the free vibrating beam eigenfunctions as follows:

$$\int_0^a X_m(x)X_i(x)\,\mathrm{d}x = \begin{cases} a & (m = i), \\ 0 & (m \neq i), \end{cases}$$

$$\int_0^b Y_n(y)Y_j(y)\,\mathrm{d}y = \begin{cases} b & (n = j), \\ 0 & (n \neq j), \end{cases}$$

$$\int_0^a \frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2X_i(x)}{\mathrm{d}x^2}\,\mathrm{d}x = \begin{cases} k_m^4/a^3 & (m = i), \\ 0 & (m \neq i), \end{cases}$$

$$\int_0^b \frac{\mathrm{d}^2Y_n(y)}{\mathrm{d}y^2}\frac{\mathrm{d}^2Y_j(y)}{\mathrm{d}y^2}\,\mathrm{d}y = \begin{cases} k_n^4/b^3 & (n = j), \\ 0 & (n \neq j). \end{cases}$$

The calculations in eqn (5) may be simplified by applying these integrations and can be analytically or numerically performed easily.

3. NUMERICAL EXAMPLES

In order to illustrate the accuracy, convergency and utility of the approach described, some numerical results are presented for two- and three-span rectangular plates with the edges simply supported and/or clamped in one and two directions and comparisons are made with previously published results where possible. The results are mainly for isotropic plates but some are for orthotropic plates. In all cases, the integrations in eqn (5) are analytically performed for the plates with opposite edges simply-supported and numerically performed for the plates with other edges by Gaussian quadrature with 16 points.

The first example treated is the two-direction, two-span plate shown in Fig. 1, which is simply supported on edges $x = 0, a$ and $y = 0, b$ and passed over intermediate line supports at $x = \alpha a$ and $y = \beta b$. Since there is only an intermediate line support in each direction, $lx = ly = 1$. From eqn (6), letting $\xi = x/a$ and $\eta = y/b$ one has

$$\varphi_m(\xi) = \sqrt{2}\sin((m+1)\pi\xi) + \sum_{k=0}^{4} C_{mk}\xi^k, \quad m = 1, 2, 3, \ldots,$$

$$\psi_n(\eta) = \sqrt{2}\sin((n+1)\pi\eta) + \sum_{k=0}^{4} D_{nk}\eta^k, \quad n = 1, 2, 3, \ldots.$$

Satisfaction of the zero deflection and zero second derivative at the simply-supported edges

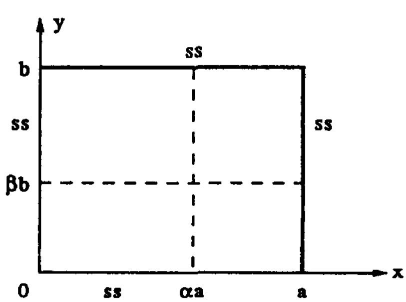

Fig. 1. Two-direction, two-span plate, simply supported all around.

and zero deflections at the intermediate line supports yields

$$C_{m0} = C_{m2} = 0, \quad C_{m1} = C_{m4} = -C_{m3}^2 = -\sqrt{2}\sin((m+1)\pi\alpha)/(\alpha^4 - 2\alpha^3 + \alpha);$$
$$D_{n0} = D_{n2} = 0, \quad D_{n1} = D_{n4} = -D_{n3}^2 = -\sqrt{2}\sin((n+1)\pi\beta)/(\beta^4 - 2\beta^3 + \beta).$$

The eigenvalue equation (5) can be easily analytically given by the use of the next integrations:

$$\int_0^1 \varphi_m(\xi)\varphi_i(\xi)\,d\xi = \delta(m-i) + \frac{24\sqrt{2}}{\pi^5}\left(\frac{C_{i1}}{(m+1)^5}(1+(-1)^m) + \frac{C_{m1}}{(i+1)^5}(1+(-1)^i)\right) + \tfrac{31}{630}C_{m1}C_{i1},$$

$$\int_0^1 \frac{d\varphi_m(\xi)}{d\xi}\frac{d\varphi_i(\xi)}{d\xi}\,d\xi = -\int_0^1 \varphi_m(\xi)\frac{d^2\varphi_i(\xi)}{d\xi^2}\,d\xi = \pi^2(m+1)(i+1)\delta(m-i) + \frac{24\sqrt{2}}{\pi^3}\left(\frac{C_{i1}}{(m+1)^3}(1+(-1)^m) + \frac{C_{m1}}{(i+1)^3}(1+(-1)^i)\right) + \tfrac{17}{35}C_{m1}C_{i1},$$

$$\int_0^1 \frac{d^2\varphi_m(\xi)}{d\xi^2}\frac{d^2\varphi_i(\xi)}{d\xi^2}\,d\xi = \pi^4(m+1)^2(i+1)^2\delta(m-i) + \frac{24\sqrt{2}}{\pi}\left(\frac{C_{i1}}{m+1}(1+(-1)^m) + \frac{C_{m1}}{i+1}(1+(-1)^i)\right) + \tfrac{24}{5}C_{m1}C_{i1},$$

where $\delta(m-i)$ is the Dirac delta function. The integrations in the y-direction may be identically obtained simply by replacing C with D, m with n and i with j.

Results for a rectangular plate of aspect ratio $a/b = 1.5$ with $\alpha = \beta = 1/3$ and for a square plate with various values of α and β are given in Table 1. All results are in close agreement with Takahasi and Chishaki (1979), Leissa (1973) and Kim and Dickinson (1987) except for a square plate with $\alpha = 1/4$, $\beta = 1/2$ where Kim and Dickinson missed the fundamental frequency of the plate, in the opinion of the author. The partial convergency

Table 1. Frequency parameters $(\rho h\omega^2 b^4/D)^{1/2}$ for the two-direction, two-span isotropic plate shown in Fig. 1

Side ratio a/b	Support locations α	β	No. of terms in series (4) $m\times n$	Mode sequence number 1	2	3	4	5	6
1	0.25	0.5	1×1	60.893					
			2×2	60.171	80.191	116.91	132.60		
			3×3	60.170	80.191	116.91	132.60	177.40	197.39
			4×4	60.070	78.676	116.54	130.13	177.14	197.39
			5×5	59.964	78.578	116.17	129.76	177.06	197.39
			Kim, 6×6		79.040	116.48	130.58	178.91	198.70
	0.1	0.1	5×5	31.642	71.273	71.498	109.96	135.72	135.61
	0.2	0.2	5×5	38.278	87.020	87.163	135.59	165.04	165.07
	0.25	0.25	5×5	42.709	97.070	97.173	151.86	177.06	177.06
			Kim, 6×6	42.844	97.437	97.531	152.58	178.59	178.59
	0.3	0.3	5×5	48.247	108.28	108.32	160.05	160.21	170.18
	0.4	0.4	5×5	63.383	105.36	105.54	145.55	156.73	156.56
	0.5	0.5	5×5	78.956	95.453	95.453	109.93	197.39	197.39
			Kim, 6×6	78.958	95.911	95.911	110.81	199.02	199.02
			Leissa	78.957	94.485	94.485	108.22	197.39	197.39
1.5	$1/\sqrt{3}$	$1/\sqrt{3}$	5×5	49.122	63.239	84.309	91.927	97.059	124.61
			Kim, 6×6	49.293	63.925	85.322	94.445	98.712	128.15
			Takahashi	49.305	62.907	83.892	91.301	96.295	123.41

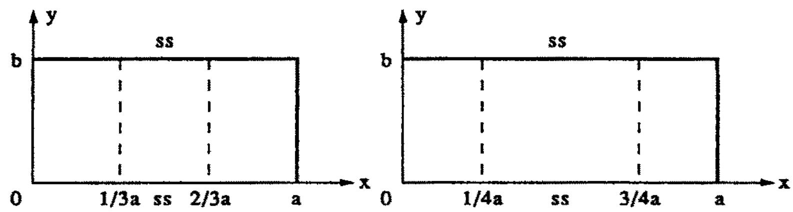

Fig. 2. One-direction, three-span plate, simply supported at two opposite edges.

study suggests that the rate of convergence is reasonably fast and the accuracy is generally better than Kim and Dickinson's (1987) (in their paper, six terms in each of the series $\varphi_m(x)$ and $\psi_n(y)$ were used) because the frequencies obtained are upper bounds by the use of the Rayleigh–Ritz method.

The second example treated is the three-span, continuous plate in one direction shown in Fig. 2. The plate is simply supported along $y = 0, b$, passes over intermediate line supports at $x = \alpha a$ and βa. In order to compare with the available results, the calculations are carried out only for a plate of aspect ratio $a/b = 3$ with $\alpha = 1/3$, $\beta = 2/3$ and for a plate of aspect ratio $a/b = 4$ with $\alpha = 1/4$, $\beta = 3/4$. Results for both edges $(x = 0, a)$ simply-supported–simply-supported (S–S), clamped–simply-supported (C–S) and clamped–clamped (C–C) are given in Table 2. The results are respectively compared with the exact solutions given by Azimi *et al.* (1984) and the Rayleigh–Ritz solutions given by Kim and Dickinson (1987). The present results are in close agreement with those results but for the C–S edges $(x = 0, a)$ and aspect ratio $a/b = 3$ with $\alpha = 1/3$, $\beta = 2/3$, where Azimi *et al.* obviously missed the second eigenfrequency of the plate by comparing with the results of the plate with the C–C edges $(x = 0, a)$ because the increase in restraint to the edges is bound to result in the increase of the values of the eigenfrequencies of the plate according to the vibration theory of structures.

As a third example, the two-direction, three-span orthotropic plate shown in Fig. 3 is treated, which passes over line supports at $x = \alpha_1 a, \beta^2 a$ and $y = \alpha_2 b, \beta_2 b$ in the x- and y-directions respectively. The first six eigenfrequency parameters for a square plate and for a plate of aspect ratio $a/b = 2$ with various boundary conditions are presented. The results in Tables 3–7 are, respectively, for the plate simply supported along edges $x = 0, a$ and

Table 2. Frequency parameters $(\rho h\omega^2 b^4/D)^{1/2}$ for the one-direction, three-span isotropic plate shown in Fig. 2, $m \times n = 5 \times 5$

Side ratio a/b	Support locations α	β	Type of edges	Mode sequence number 1	2	3	4	5	6
3	1/3	2/3	S–S	19.739	21.854	26.373	49.348	49.348	50.876
			Azimi‡	19.74	21.60	26.00	49.35		
			C–S	20.269	23.843	28.523	49.712	50.848	52.029
			Azimi‡	20.22		28.07	49.63	50.69	
			C–C	21.611	26.235	29.507	50.545	53.661	55.206
			Azimi‡	21.60	26.00	28.95	50.45		
4	1/4	3/4	S–S	12.968	19.739	21.794	24.303	36.099	42.341
			Kim†	12.940	19.739	21.639	23.914	35.529	42.289
			Azimi‡	12.92	19.74	21.53	23.65	35.21	42.24
			C–S	12.989	20.130	23.025	27.103	36.716	42.341
			Azimi‡	12.94	20.10	22.64	26.50	35.59	42.24
			C–C	13.011	20.889	25.838	27.829	37.371	42.342
			Kim†	12.982	20.834	25.746	27.328	36.440	42.296
			Azimi‡	12.96	20.81	25.64	27.12	35.97	42.25

†$m \times n = 6 \times 6$.
‡Exact solution.

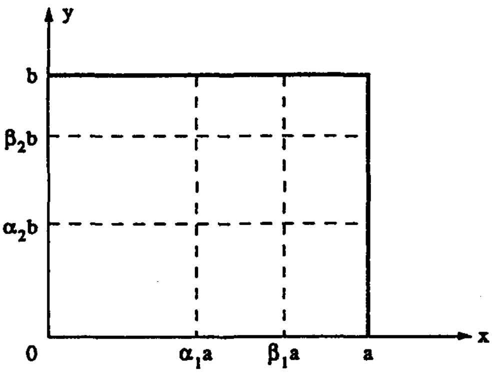

Fig. 3. Two-direction, three-span plate.

$y = 0, b$ (SSSS), clamped–simply-supported along $x = 0, a$ and $y = 0, b$ (CCSS), clamped–clamped along $x = 0, a$ and $y = 0, b$ (CCCC), clamped–clamped along $x = 0, a$ and simply-supported–simply-supported along $y = 0, b$ (CSCS), clamped–clamped along $x = 0, a$ and clamped–simply-supported along $y = 0, b$ (CCCS). The convergency may be seen to be very fast and the accuracy is better than Kim and Dickinson's (1987) for both isotropic and orthotropic plates.

4. CONCLUDING REMARKS

It may be seen that the use of a combination of beam eigenfunctions and polynomials in the Rayleigh–Ritz method is applicable to the study of a variety of one- and two-direction continuous plate problems. The beam eigenfunctions which satisfy the boundary conditions of the plate are selected as the main parts of the admissible functions and the polynomials

Table 3. Frequency parameters $(\rho h\omega^2 b^4/D)^{1/2}$ for the two-direction, three-span isotropic plate shown in Fig. 3 (SSSS)

Side ratio	Support locations				No. of terms in series (4)	Mode sequence number					
a/b	α_1	β_1	α_2	β_2	$m\times n$	1	2	3	4	5	6
1	0.3	0.65	0.45	0.85	1×1	140.54					
					2×2	139.65	171.65	185.35	214.35		
					3×3	139.53	170.79	176.61	204.57	219.25	248.95
					4×4	139.12	170.28	176.03	203.84	214.10	243.57
					5×5	139.09	169.04	175.60	202.06	210.78	239.84
	0.1	0.85	0.2	0.7	1×1	87.570					
					2×2	84.917	145.51	160.75	222.28		
					3×3	82.846	144.17	155.72	217.64	226.58	238.79
					4×4	82.694	142.01	155.40	214.93	222.75	236.74
					5×5	82.501	141.23	155.31	214.18	221.42	235.59
	1/3	3/4	1/2	2/3	1×1	126.19					
					2×2	125.66	173.88	195.06	240.68		
					3×3	121.94	164.40	183.58	222.66	243.48	250.660
					4×4	121.46	164.18	182.89	222.41	238.67	249.75
					5×5	120.85	162.78	180.74	219.23	237.81	245.82
	1/10	1/3	3/20	2/3	1×1	93.790					
					2×2	81.237	138.19	150.95	209.09		
					3×3	80.500	136.56	147.17	203.77	217.24	254.12
					4×4	80.358	136.50	146.93	203.63	216.89	252.29
					5×5	80.290	135.92	146.74	202.70	214.59	252.04
2	1/4	3/4	1/4	3/4	5×5	71.509	94.974	102.19	111.16	169.36	197.39
	1/3	2/3	1/3	2/3	5×5	111.03	114.47	120.97	134.66	137.69	143.35
	1/3	2/3	1/4	3/4	5×5	83.286	90.543	140.79	151.72	180.12	183.20
	1/4	2/3	1/3	3/4	5×5	92.556	101.25	119.71	137.33	142.95	145.12

Table 4. Frequency parameters $(\rho h\omega^2 b^4/H)^{1/2}$ for the two-direction, three-span orthotropic plate shown in Fig. 3 (CCSS)

Material properties $D_x/H; D_y/H$	Side ratio a/b	Support locations α_1	β_1	α_2	β_2	No. of terms in series (4) $m \times n$	Mode sequence number 1	2	3	4	5	6
1 ; 1	1	0.35	0.7	0.35	0.7	1×1	192.61					
						2×2	192.26	228.36	228.68	261.77		
						3×3	190.86	227.65	227.87	256.54	256.58	261.31
						4×4	190.72	226.66	226.94	256.43	256.44	259.45
						5×5	189.95	226.25	226.53	251.86	251.98	259.42
						Kim, 6×6	190.69	226.87	227.18	259.99	265.88	265.93
		0.25	0.6	0.35	0.7	5×5	164.85	203.34	211.09	231.09	244.92	268.81
		0.4	0.75	0.3	0.65	5×5	173.61	207.45	207.76	237.98	279.98	286.05
	2	0.35	0.7	0.35	0.7	5×5	120.59	126.96	131.49	163.68	168.95	172.96
		0.25	0.6	0.35	0.7	5×5	114.94	123.68	156.13	158.80	163.49	166.19
		0.4	0.75	0.3	0.65	5×5	107.32	113.84	133.88	150.67	155.95	163.01
1.543 ; 4.810†	1	0.35	0.7	0.35	0.7	5×5	281.60	317.33	345.32	370.85	399.86	422.55
	2	0.35	0.7	0.35	0.7	5×5	229.23	233.59	236.64	277.70	297.20	328.30

†The values given for plywood by Hearmon (1959).

with undetermined coefficients are added as the modified parts of the admissible functions in order to satisfy the zero deflection conditions at intermediate line supports. The coefficients in the polynomials are decided by the boundary conditions and the intermediate line supports of the plate. The starting order of beam eigenfunctions and the number of the terms of polynomials in each direction are dependent on the number of intermediate line supports in that direction in order to obtain the right results. The integrations of multiplication of beam eigenfunctions with polynomials may be numerically performed or be analytically written in recurrent form without any approximate calculation, if one wants. This presents a significant advantage over many other methods of analysis of plate systems where the complexity and computation cost often increase substantially with increasing number of spans. The proposed method is only valid for uniform thickness plates.

Table 5. Frequency parameters $(\rho h\omega^2 b^4/H)^{1/2}$ for the two-direction, three-span orthotropic plate shown in Fig. 3 (CCCC)

Material properties $D_x/H; D_y/H$	Side ratio a/b	Support locations α_1	β_1	α_2	β_2	No. of terms in series (4) $m \times n$	Mode sequence number 1	2	3	4	5	6
1 ; 1	1	0.35	0.7	0.35	0.7	1×1	201.19					
						2×2	201.19	249.91	249.97	293.35		
						3×3	199.59	242.51	242.55	279.79	299.42	299.42
						4×4	198.76	242.08	242.13	279.76	291.54	291.58
						5×5	198.04	240.01	240.03	276.61	291.09	291.15
						Kim, 4×4	202.40	249.31	249.47	290.71	344.19	344.38
						Kim, 5×5	199.07	244.48	244.54	284.09	299.96	299.96
						Kim, 6×6	198.55	243.27	243.32	282.31	297.20	297.20
		0.25	0.6	0.35	0.7	5×5	187.54	223.01	230.53	261.23	281.59	309.34
		0.4	0.75	0.3	0.65	5×5	187.32	222.38	231.43	261.65	282.23	309.32
	2	0.35	0.7	0.35	0.7	5×5	127.90	134.24	145.18	180.39	185.29	193.20
		0.25	0.6	0.35	0.7	5×5	125.07	131.06	161.14	178.05	181.24	182.78
		0.4	0.75	0.3	0.65	5×5	124.47	130.59	163.95	178.60	179.24	184.02
1.543 ; 4.810	1	0.35	0.7	0.35	0.7	1×1	305.48					
						2×2	304.96	352.39	423.65	460.69		
						3×3	301.61	344.54	402.52	413.55	446.46	494.08
						4×4	301.05	344.24	394.66	413.33	446.45	487.71
						5×5	300.30	342.08	394.16	409.21	441.42	484.22
						Kim, 6×6	301.03	345.36	401.76	414.92	449.04	494.99
		0.25	0.6	0.35	0.7	5×5	291.76	325.71	402.45	428.33	469.86	516.73
		0.4	0.75	0.3	0.65	5×5	290.73	324.10	405.13	430.37	468.42	522.24
	2	0.35	0.7	0.35	0.7	5×5	247.56	251.97	259.25	295.08	317.35	369.08
		0.25	0.6	0.35	0.7	5×5	246.02	249.84	270.91	285.21	308.44	367.88
		0.4	0.75	0.3	0.65	5×5	244.64	248.51	271.24	282.71	306.16	370.73

Table 6. Frequency parameters $(\rho h\omega^2 b^4/H)^{1/2}$ for the two-direction, three-span orthotropic plate shown in Fig. 3 (CSCS)

Material properties D_x/H; D_y/H	Side ratio a/b	Support locations α_1	β_1	α_2	β_2	No. of terms in series (4) $m\times n$	Mode sequence number 1	2	3	4	5	6
1 ; 1	1	0.35	0.7	0.35	0.7	1×1	186.72					
						2×2	185.83	215.13	236.94	263.11		
						3×3	184.86	213.29	231.09	254.79	257.92	288.36
						4×4	184.43	212.65	230.05	252.31	254.59	280.87
						5×5	184.05	210.74	228.25	248.46	251.37	280.80
						Kim, 6×6	184.34	212.77	231.39	256.24	262.44	286.99
		0.25	0.6	0.35	0.7	5×5	173.09	200.12	210.60	234.53	238.43	269.25
		0.4	0.75	0.3	0.65	5×5	173.20	200.18	210.14	234.19	238.44	269.02
	2	0.35	0.7	0.35	0.7	5×5	108.37	115.61	127.68	140.91	147.03	157.96
		0.25	0.6	0.35	0.7	5×5	105.21	112.01	138.01	143.92	145.04	166.17
		0.4	0.75	0.3	0.65	5×5	105.18	112.08	137.97	144.00	148.30	163.84
1.543 ; 4.810	1	0.35	0.7	0.35	0.7	5×5	261.10	307.93	326.45	364.34	365.85	416.13
	2	0.35	0.7	0.35	0.7	5×5	200.11	205.44	214.02	255.03	275.43	279.59

Table 7. Frequency parameters $(\rho h\omega^2 b^4/H)^{1/2}$ for the two-direction, three-span orthotropic plate shown in Fig. 3 (CCCS), $m\times n = 5\times 5$

Material properties D/H; D/H	Side ratio a/b	Support locations α_1	β_1	α_2	β_2	Mode sequence number 1	2	3	4	5	6
1 ; 1	1	0.35	0.7	0.35	0.7	193.78	229.59	237.58	254.80	268.75	288.79
		0.25	0.6	0.35	0.7	183.33	219.00	219.56	245.53	251.65	274.90
		0.4	0.75	0.3	0.65	174.32	208.32	211.60	241.42	280.36	307.93
	2	0.35	0.7	0.35	0.7	121.07	127.68	138.10	164.08	169.42	178.41
		0.25	0.6	0.35	0.7	117.87	124.38	159.30	161.19	166.71	174.40
		0.4	0.75	0.3	0.65	107.38	114.03	146.55	150.71	156.05	167.45
1.543 ; 4.810	1	0.35	0.7	0.35	0.7	285.68	330.65	373.92	384.82	410.15	444.25
	2	0.35	0.7	0.35	0.7	229.57	234.35	241.69	281.46	305.05	328.55

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